Math 251 Midterm 2 Sample

Name: ____________________

This exam has 8 questions, for a total of 100 points.

Please answer each question in the space provided. You need to write **full solutions**. Answers without justification will not be graded. Cross out anything the grader should ignore and circle or box the final answer.

Question	Points	Score
1	10	
2	14	
3	14	
4	10	
5	10	
6	10	
7	12	
8	20	
Total:	100	

Question 1. (10 pts)

$$f(x,y,z) = z\sqrt{y^2 - x}$$

(a) Find the gradient of the function $f(x,y,z)$

Solution:

$$\begin{aligned}\nabla f &= \langle f_x, f_y, f_z \rangle \\ &= \left\langle -\frac{z}{2}(y^2 - x)^{-1/2}, zy(y^2 - x)^{-1/2}, \sqrt{y^2 - x} \right\rangle\end{aligned}$$

(b) Find the maximum rate of change of $f(x,y,z)$ at the point $(5,3,1)$.

Solution: The maximum rate of change at $(5,3,1)$ is

$$\|\nabla f(5,3,1)\| = \|\langle \frac{-1}{4}, \frac{3}{2}, 2 \rangle\| = \frac{\sqrt{101}}{4}$$

Question 2. (14 pts)

Given

$$f(x, y) = x^2y - x^2 - y^2$$

Determine all local maximum, minimum and saddle points.

Solution: First, find all critical points. We need to solve

$$\begin{cases} f_x = 2xy - 2x = 0 \\ f_y = x^2 - 2y = 0 \end{cases}$$

Case 1: $x = 0$, then from the second equation, we see

$$y = 0$$

So $(0, 0)$ is a critical point.

case 2: $x \neq 0$, then the first equation gives $y = 1$. Now the second equation implies that $x^2 = 2$, so $x = \pm\sqrt{2}$. We have critical points

$$(\pm\sqrt{2}, 1)$$

Use the second derivative test,

$$f_{xx} = 2y - 2$$

$$f_{yy} = -2$$

$$f_{xy} = 2x$$

So

$$D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = 4 - 4y - 4x^2$$

(1) At $(0, 0)$, $D(0, 0) = 4$ and $f_{xx}(0, 0) = -2$, so $(0, 0)$ is a local max.

(2) At $(\pm\sqrt{2}, 1)$, $D(\pm\sqrt{2}, 1) = -8 < 0$, so $(\pm\sqrt{2}, 1)$ are saddle points.

Question 3. (14 pts)

Use the Lagrange multiplier method to find the absolute extreme values of the function

$$f(x, y) = xy$$

with the constraint $x^2 + 4y^2 = 8$.

Solution: The constraint is

$$g(x, y) = x^2 + 4y^2 - 8 = 0$$

Apply the Lagrange multiplier method,

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x, y) = 0 \end{cases}$$

That is,

$$\begin{cases} y = \lambda(2x) \\ x = \lambda(8y) \\ x^2 + 4y^2 - 8 = 0 \end{cases}$$

Using the first two equations, we get

$$16\lambda^2 = 1.$$

So $\lambda = \pm(1/4)$.

Case 1: If $\lambda = 1/4$, then $x = 2y$. Use the third equation to solve for x and y. We get two solutions

$$(x, y) = (2, 1) \text{ or } (-2, -1)$$

Case 2: If $\lambda = -(1/4)$, then $x = -2y$. Use the third equation to solve for x and y. We get two solutions

$$(x, y) = (-2, 1) \text{ or } (2, -1)$$

So the absolute max is $f(2, 1) = f(-2, -1) = 2$, and the absolute min is $f(-2, 1) = f(2, -1) = -2$.

Question 4. (10 pts)

Rewrite (but do not evaluate)

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} (x+y)dydx$$

in polar coordinates.

Solution:

Notice that $y = \sqrt{2x - x^2}$ gives

$$x^2 + y^2 = 2x$$

which is the circle centered at $(1, 0)$ with radius 1. In polar coordinates, we have

$$r^2 = 2r\cos\theta \Longrightarrow r = 2\cos\theta$$

And the other bound $y = 0$ tells us that the region is above y-axis.

Now x goes from 0 to 2 tells us that the region is actual the entire upper half of the disk.

$$\int_0^{\pi/2} \int_0^{2\cos\theta} (r\cos\theta + r\sin\theta)r\ drd\theta$$

Question 5. (10 pts)

Rewrite (but do not evaluate)

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} (x+z)\ dzdxdy$$

in spherical coordinates.

Solution: From the bounds, we can see that the region is exactly everything between the cone $z = \sqrt{x^2+y^2}$ and the upper hemisphere $z = \sqrt{2-x^2-y^2}$.

$$\int_{0}^{2\pi}\int_{0}^{\pi/4}\int_{0}^{\sqrt{2}} (\rho\sin\varphi\cos\theta + \rho\cos\varphi)\rho^2\sin\varphi\ d\rho d\varphi d\theta$$

Question 6. (10 pts)

Evaluate the following integral by switching the order of integration.

$$\int_0^1 \int_{\sqrt{y}}^1 e^{x^3}\ dxdy$$

Solution:

$$\int_0^1 \int_{\sqrt{y}}^1 e^{x^3}\ dxdy$$

$$= \int_0^1 \int_0^{x^2} e^{x^3} dydx$$

$$= \int_0^1 e^{x^3} x^2 dx = \cdots$$

use substitution $u = x^3$.

$$= \left. \frac{e^{x^3}}{3} \right|_{x=0}^{x=1} = \frac{e-1}{3}$$

Question 7. (12 pts)

Find the area of the region above the circle $x^2 + y^2 = 4y$ and below the circle $x^2 + y^2 = 4$.

Solution: use polar coordinates. We shall write the given equations in polar coordinates.

$$x^2 + y^2 = 4y \quad \Rightarrow \quad r^2 = 4r\sin\theta, \text{ i.e. } r = 4\sin\theta$$

$$x^2 + y^2 = 4 \quad \Rightarrow \quad r = 2$$

We need to use these two equations to find out the bounded for r and θ. You should certainly draw a picture of this region first. Set them equal to each other and we get

$$4\sin\theta = 2$$

Therefore, $\theta = \pi/6$ or $5\pi/6$.

$$\begin{aligned} \text{Area} &= \iint_D dA \\ &= \int_{\pi/6}^{5\pi/6} \int_0^2 r dr d\theta + \int_0^{\pi/6} \int_0^{4\sin\theta} r dr d\theta + \int_{5\pi/6}^{\pi} \int_0^{4\sin\theta} r dr d\theta \end{aligned}$$

or equivalently, by symmetry,

$$\begin{aligned} \text{Area} &= \iint_D dA \\ &= 2\int_{\pi/6}^{\pi/2} \int_0^2 r dr d\theta + 2\int_0^{\pi/6} \int_0^{4\sin\theta} r dr d\theta \\ &= \frac{4\pi}{3} + \frac{4\pi}{3} - 2\sqrt{3} \\ &= \frac{8\pi}{3} - 2\sqrt{3} \end{aligned}$$

Question 8. (20 pts)

E is the solid that is between the upper half of the sphere $x^2+y^2+z^2=4$ and the cone $z=\sqrt{x^2+y^2}$.

(a) Write the volume of E as a triple integral in xyz-coordinates.

Solution: Solve z from $x^2+y^2+z^2=4$ to get the upper bound for z. The circle of intersection between the cone and the sphere is $x^2+y^2=2$ at $z=\sqrt{2}$. We can get this by plugging $z=\sqrt{x^2+y^2}$ into the equation $x^2+y^2+z^2=4$. From this we know that, the projection of the solid E to the xy-plane is the disk centered at $(0,0)$ with radius $\sqrt{2}$.

$$\int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} dzdxdy$$

(b) Write the volume of E as a triple integral in cylindrical coordinates.

Solution:

$$\int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}} r\ dzdrd\theta$$

(c) Write the volume of E as a triple integral in spherical coordinates.

Solution:

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^2 \rho^2 \sin\varphi \; d\rho d\varphi d\theta$$

(d) Use one of your answers from part (a), (b) and (c) to calculate the volume of E.

Solution: Use spherical coordinates

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^2 \rho^2 \sin\varphi \; d\rho d\varphi d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{\rho^3}{3}\Big|_{\rho=0}^{\rho=2} \sin\varphi \; d\varphi d\theta$$
$$= \frac{8}{3}(1 - \frac{\sqrt{2}}{2})2\pi$$